Partial degeneracy breaking of the hydrogen energy spectrum from $su_q(2)$

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Abstract

In this work we investigate the q-deformation of the $so(4)$ dynamical symmetry of the hydrogen atom using the theory of the quantum group $su_q(2)$, and construct the discrete part of the energy spectrum. This will lead to a partial breaking of the degeneracy of the energy levels and to a reduction of the Hilbert space.

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1 Introduction

In this work we investigate the construction of the spectrum of the hydrogen atom using the $su_q(2)$ quantum group. Replacing the original algebra with its deformed version, it is still possible to use the canonical approach to find the energy levels, which now depend on one extra quantum number. A new constraint arising as a consequence of the orthogonality between the Runge-Lenz vector and the angular momentum leads to a smaller Hilbert space.

Quantum groups and deformations have gained a prominent role in physics, with applications on a great variety of subjects, e.g. quantum field theory on noncommutative spaces [1, 2]. In particular they are the mathematical tool used to investigate the idea of a minimal length [3, 4]. For this reason, much attention has been devoted to looking into deformed symmetries in physical systems, such as the deformed rotational symmetry $su_q(2)$ of the harmonic oscillator [5]. In some references the q-deformation is made at the level of the Schrödinger equation [6, 7]. The hydrogen atom on a noncommutative curved spacetime has been recently investigated in [8].

Another possibility lies in the investigation of deformed dynamical symmetries. The best known example is the relation between $so(4)$ and the spectrum of the hydrogen atom, achieved by using the Runge-Lenz vector, which enlarges the symmetry group. The spectrum is then given by the restriction of the representation (i, j) to (i, i). The q-deformation of the dynamical symmetry of the hydrogen atom has been attempted in many different ways, either through the separation $so(4) = su(2) \oplus su(2)$ [9, 10, 11, 12], or through the use of the Kustaanheimo-Stiefel transformation [13, 14].

The structure of this paper is as follows: we first briefly review the derivation of the spectrum of the hydrogen atom using the $so(4)$ dynamical symmetry. We then present how the q-deformation changes the energy spectrum, showing the new dependence on the quantum numbers and the reduction of the Hilbert space. We finish by discussing the partial breaking of the degeneracy of the energy levels.

2 Undeformed case

We start by briefly reviewing the derivation of the spectrum of the hydrogen atom using its dynamical symmetry $so(4)$ [15, 16, 17].

We want to solve the Schrödinger equation for the Coulomb Hamiltonian

$$H = \frac{p^2}{2\mu} - \frac{k}{r}, \tag{1}$$

where μ is the reduced mass, $k = e^2/4\pi\epsilon_0$ and e the charge of the electron.

It is known from the Kepler problem that the vectors $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{M} = \mathbf{v} \times \mathbf{L} - k\left(\frac{\mathbf{r}}{r}\right)$ are constants of motion. They are, respectively, the angular momentum and the Runge-Lenz vector.

Now using the correspondence principle and taking the Hermitian part of $\mathbf{M}$, one finds that the commutation relations among L_i, M_i and H are

$$\begin{aligned}
[L_i, H] &= [M_i, H] = 0 && (2)\\
[L_i, L_j] &= i\hbar\epsilon_{ijk}L_k && (3)\\
[L_i, M_j] &= i\hbar\epsilon_{ijk}M_k && (4)\\
[M_i, M_j] &= -\frac{2i\hbar}{\mu}\epsilon_{ijk}L_k H, && (5)
\end{aligned}$$

and, from the commutation relations between x_i and p_i, that the relations

$$\mathbf{L}\cdot\mathbf{M} = \mathbf{M}\cdot\mathbf{L} = 0, \quad (6)$$

$$\mathbf{M}^2 = k^2 + \frac{2}{\mu}(\mathbf{L}^2 + \hbar^2)H \quad (7)$$

hold.

If we now restrict ourselves to the Hilbert subspace corresponding to a negative eigenvalue E of H, we can rescale $\mathbf{M}$ as

$$\tilde{\mathbf{M}} = \sqrt{-\frac{\mu}{2E}}\mathbf{M}, \quad (8)$$

and therefore introduce the operators

$$\begin{aligned}
\mathbf{I} &= \frac{\mathbf{L} + \tilde{\mathbf{M}}}{2} && (9)\\
\mathbf{J} &= \frac{\mathbf{L} - \tilde{\mathbf{M}}}{2} && (10)
\end{aligned}$$

which form two disjoint sets of operators satisfying the $su(2)$ algebra:

$$[I_z, I_\pm] = I_\pm, \qquad [I_+, I_-] = 2I_z \quad (11)$$

$$[J_z, J_\pm] = J_\pm, \qquad [J_+, J_-] = 2J_z, \quad (12)$$

so that we have in principle 4 quantum numbers, i, m and j, p, diagonalizing $\mathbf{I}^2, I_z$ and $\mathbf{J}^2, J_z$, respectively.

Relating these 2 algebras to the hydrogen atom means rewriting relations (6) and (7) in terms of $\mathbf{I}$ and $\mathbf{J}$ as

$$(\mathbf{I}^2 - \mathbf{J}^2)|i, m, j, p\rangle = 0, \quad (13)$$

and

$$(\mathbf{I}^2 + \mathbf{J}^2)|i, m, j, p\rangle = -\left(\frac{\mu k^2}{4E} + \frac{\hbar^2}{2}\right)|i, m, j, p\rangle. \tag{14}$$

Since

$$\mathbf{I}^2|i, m, j, p\rangle = i(i+1)\hbar^2|i, m, j, p\rangle \tag{15}$$
$$\mathbf{J}^2|i, m, j, p\rangle = j(j+1)\hbar^2|i, m, j, p\rangle, \tag{16}$$

relation (13) yields that $i = j$, and, as a result of (14), we have that

$$2\hbar^2 j(j+1) = -\left(\frac{\mu k^2}{4E} + \frac{\hbar^2}{2}\right), \tag{17}$$

leading to

$$E = \frac{-\mu k^2}{2\hbar^2(2j+1)^2}, \tag{18}$$

so that we can identify $2j+1 = n$ with the principal quantum number of the hydrogen atom.

Since $i = j$, the Hilbert space is of the appropriate size.

It is important to note that j can take half-integer values, and that I_z and J_z can each have $2j+1$ independent eigenvalues, so that the degeneracy of a given state is $(2j+1)^2 = n^2$ as expected.

3 Deformed Spectrum

We choose to deform the $so(4) = su(2) \oplus su(2)$ symmetry by a parameter q by means of the very well known theory of $su_q(2)$ [18, 19, 20, 21], where the commutation relations are written as

$$[I_z, I_\pm] = I_\pm, \qquad [I_+, I_-] = 2[I_z] \tag{19}$$
$$[J_z, J_\pm] = J_\pm, \qquad [J_+, J_-] = 2[J_z], \tag{20}$$

where

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh(sx)}{\sinh s}, \tag{21}$$

with $s = \ln q$ and q a real parameter. In this section, we take $\hbar = 1$.

As for the representation, this means that I_z and J_z will act exactly like before, while for the other operators we have

$$I_\pm|i, m, j, p\rangle = \sqrt{[i \pm m + 1][i \mp m]}|i, m \pm 1, j, p\rangle, \tag{22}$$

and analogously for $J_\pm$, so that

$$I_\pm I_\mp |i,m,j,p\rangle = ([i][i+1] - [m][m \mp 1])|i,m,j,p\rangle, \tag{23}$$

which is still diagonal in the old Hilbert space (i.e., using the same basis).

Using these expressions, it is easy to deform relations (13) and (14) and see that $(\mathbf{I}^2 - \mathbf{J}^2)|i,m,j,p\rangle = 0$ implies that $i = j$ and $m^2 = p^2$. Note that the second restriction is completely new to the deformed case.

From (14) we have that

$$(\mathbf{I}^2 + \mathbf{J}^2)|i,m,j,p\rangle = \left(2[j][j+1] - [m]([m+1] + [m-1]) + 2m^2\right)|i,m,j,p\rangle. \tag{24}$$

This immediately gives the deformed energy for the hydrogen atom:

$$E_{jm} = \frac{-\mu k^2}{8[j][j+1] - 4[m]([m+1] + [m-1]) + 8m^2 + 2}. \tag{25}$$

It easy to realize that this has the correct limit when $q \to 1$.

3.1 Degeneracy breaking

Since the energy E_{jm} now depends additionaly on the quantum number m, part of the degeneracy of the energy spectrum is broken. It is easy to see that changing $m \to -m$ leaves the energy invariant, and thus for each j there are $j+1$ possible energy levels.

Let us now work out the degeneracy of each of these levels. Because the result does not depend on p, one could expect the usual $2j+1$ degeneracy that comes from it, but this is not the case due to the constraint $(\mathbf{I}^2 - \mathbf{J}^2)|i,m,j,p\rangle = 0$, which makes $m^2 = p^2$ and thus $p = \pm m$. This means that E_{jm} is four-fold degenerate for $m \neq 0$ and nondegenerate for $m = 0$, and therefore each value of j corresponds to $4j+1$ states, as opposed to the usual $(2j+1)^2$.

4 Conclusion

In this work we constructed a q-deformed version of the energy spectrum of the hydrogen atom by deforming the commutation relations of the $so(4)$ dynamical symmetry by means of the $su_q(2)$ quantum group. In this setting, the actions of $I_\pm$ $J_\pm$ are analogous to the undeformed ones, replacing the coefficients by q-numbers. This enables us to use the eigenvalues of $\mathbf{I}^2$ and $\mathbf{J}^2$ and the relations among them to find the discrete part of the spectrum, which now depends on two quantum numbers.

This leads to a partial breaking of the degeneracy of the energy levels, due to the fact that, although $\mathbf{I}^2$ and $\mathbf{J}^2$ are still diagonal, their eigenvalues have an extra dependence on m and p. The degeneracy is not completely removed because of the constraint $m = \pm p$ and the invariance of the energy under $m \to -m$. The residual degeneracy does not depend on j. The number of states corresponding to each value of the quantum number j is smaller than in the usual case.

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